（第 1 頁共 2 頁）（限用答案本作答）　　　　　　□可使用計算機 ☑不可使用計算機

## 離散數學(50%)

1. Let $A = B = C = R$ (*real numbers*), and let $f : A \to B$, $g : B \to C$ be defined by $f(a) = a+1$ and $g(b) = b^2 + 2$. Find (10%)
   (a). $(g \circ f)(-2)$
   (b). $(f \circ g)(-2)$
   (c). $(g \circ f)(x)$
   (d). $(f \circ g)(x)$
   (e). $(f \circ f)(y)$

2. Let $G$ be the set of all nonzero real numbers and let $a*b = (ab)/2$. Show that $(G, *)$ is an Abelian group. (10%)

3. Let $(S_1, *_1)$, $(S_2, *_2)$, and $(S_3, *_3)$ be semigroups, and $f : S_1 \to S_2$ and $g : S_2 \to S_3$ be homomorphisms. Prove that $g \circ f$ is a homomorphism from $S_1$ to $S_3$. (5%)

4. Let $S = (1 \times 2$ matrices, $\square)$, where $[x, y] \ \square \ [w, z] = [x+w, (y+z)/2]$. Determine which of the following properties hold for this structure. (15%)
   (a). Closure
   (b). Commutative
   (c). Associative

5. Find an explicit formula for the sequence defined by $c_n = 3c_{n-1} - 2c_{n-2}$ with initial conditions $c_1 = 5$ and $c_2 = 3$. (10%)

## 線性代數(50%)

6. Let **A** be a matrix defined below

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$$

Compute $\mathbf{A}^{-1000}$. (5%)

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7. Let $\mathbf{A}$ be a matrix defined below

$$\mathbf{A} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$$

Find the matrices $\mathbf{P}$ and $\mathbf{D}$ such that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$.　　(10%)

8. Find the **QR**-decomposition of

$$\mathbf{A} = \begin{bmatrix} -1 & 2 \\ 0 & -1 \\ 1 & 2 \end{bmatrix}$$　　(10%)

9. Let $\mathbf{A}$ be a matrix defined below

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -2 & 0 \\ 2 & 6 & -5 & -2 \\ 2 & 6 & 0 & 8 \end{bmatrix}$$

(a). Find a basis for the nullspace of $\mathbf{A}$.　　(6%)

(b). Find the rank and nullity of $\mathbf{A}$.　　(4%)

10. Let $T : R^2 \rightarrow R^3$ be the linear transformation defined by

$$T\left( \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_2 \\ 5x_1 - 6x_2 \\ 5x_1 + 6x_2 \end{bmatrix}$$

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Find the matrix $[T]_{\mathbf{B'},\mathbf{B}}$ for the transformation $T$ with respect to the

bases $\mathbf{B} = \{\mathbf{u}_1, \mathbf{u}_2\}$ for $R^2$ and $\mathbf{B'} = \{\mathbf{u}_1', \mathbf{u}_2', \mathbf{u}_3'\}$ for $R^3$, where

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad \mathbf{u}_1' = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{u}_2' = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}, \quad \text{and} \quad \mathbf{u}_3' = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$　　(15%)

試題完